https://www.linkedin.com/feed/update/urn:li:activity:6506432713422049280

4306. Proposed by Marius Drăgan.

Prove that

$$\left[\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}\right]=\left[\sqrt{16n+20}\right]$$

for all $n\in\mathbb{N}$.

Solution by Arkady Alt , San Jose, California, USA.

Let $R:=\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}$. Then by Cauchy Inequality we obtain

$R=1\cdot\sqrt{n}+1\cdot\sqrt{n+1}+1\cdot\sqrt{n+2}+1\cdot\sqrt{n+3}\le 2\cdot\sqrt{n+n+1+n+2+n+3}=\sqrt{16n+24}$

We have $R^2:=\left(\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}\right)^2=$

$4n+6+2\sqrt{n^2+n}+2\sqrt{n^2+2n}+2\sqrt{n^2+3n}+2\sqrt{n^2+3n+2}+2\sqrt{n^2+4n+3}+2\sqrt{n^2+5n+6}$.

Note that

$\sqrt{n^2+n}+\sqrt{n^2+2n}>2n+1\iff 2n^2+3n+2\sqrt{n^2+n}\cdot\sqrt{n^2+2n}>4n^2+4n+1\iff$

$2\sqrt{n^2+n}\cdot\sqrt{n^2+2n}>2n^2+n+1\iff 4n^4+12n^3+8n^2>4n^4+4n^3+5n^2+2n+1\iff$

$8n^3+3n^2>2n+1$ and $\sqrt{n^2+3n+2}+\sqrt{n^2+4n+3}>2n+3\iff$

$2n^2+7n+5+2\sqrt{n^2+3n+2}\cdot\sqrt{n^2+4n+3}>4n^2+12n+9\iff$

$2\sqrt{n^2+3n+2}\cdot\sqrt{n^2+4n+3}>2n^2+5n+4\iff 4(n^2+3n+2)(n^2+4n+3)>(2n^2+5n+4)^2$

$8n^3+27n^2+28n+8>0.$

Also note that $\sqrt{n^2+3n}>n+1$ and $\sqrt{n^2+5n+6}>n+2$

Hence, $R^2>4n+6+2(2n+1+n+1+2n+3+n+2)=16n+20\iff\sqrt{16n+20}<R$.

So, $\sqrt{16n+20}<R<\sqrt{16n+24}$ and remains to prove that $\sqrt{16n+24}<\left[\sqrt{16n+20}\right]+1$.

Let $m:=\left[\sqrt{16n+20}\right]+1$. Then $\sqrt{16n+20}<m\iff 16n+20<m^2\iff 16n+21\le m^2$.

But $16n+21\neq m^2$ because $16n+21\equiv 5(\operatorname{mod}16)$ and 5 is quadratic non-residue* $\operatorname{mod}16$.

Hence, $16n+21<m^2\Rightarrow 16n+22\le m^2$ and since $16n+22\equiv 6(\operatorname{mod}16)$ and 6 is quadratic

non-residue by $\operatorname{mod}16$ as well we obtain $16n+22<m^2\Rightarrow 16n+23\le m^2$.

By the same reason as above we obtain

$16n+23\le m^2\Rightarrow 16n+23<m^2\Rightarrow 16n+24\le m^2\Rightarrow 16n+24<m^2$ because 7 and 8 are quadratic non-residues by $\operatorname{mod}16$. Since

$(m-1)^2\le 16n+20<R^2<16n+24<m^2$ then

$m-1<\sqrt{16n+24}<m\iff[R]=\left[\sqrt{16n+24}\right]=\left[\sqrt{16n+20}\right]$.

* Set of quadratic residues $\operatorname{mod}16$ is $\{0,1,4,9\}$.

****